Brownian motion and Stochastic Calculus
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Assignment 0

Exercise 1

Let X_i, $i \in \mathbb{N}$ be i.i.d. with $X_i \sim \mathcal{N}(0, \sigma^2)$ for some $\sigma^2 > 0$. Let us define $Y_0 = 1$ and

$$Y_n = \exp\left(\sum_{i=1}^{n} X_i - n\frac{\sigma^2}{2}\right), \quad n \in \mathbb{N}.$$

1) Show that $(Y_n)_n$ is a martingale.
2) Snow that $Y_n \longrightarrow 0$, $\mathbb{P}-$a.s. for $n \to \infty$.
3) Is the process $(Y_n)_n$ uniformly integrable? Why/why not?
4) Assume now that $X_i \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$. For which values of $(\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$ does point 2) still hold?

Exercise 2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a (discrete) filtration $\mathbb{F} = (\mathcal{F}_n, n \in \mathbb{N} \cup \{0\})$. Let τ_n be a stopping time for every $n \in \mathbb{N}$. Which of the following are always stopping times?

1) $\sup_{n\in\mathbb{N}} \tau_n$.
2) $\inf_{n\in\mathbb{N}} \tau_n$.

Exercise 3

Let X_i, $i \in \mathbb{N}$ be i.i.d. random variables with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. Let us set $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N}.$$

(i.e., $(S_n)_n$ is a simple symmetric random walk.) Let us have two finite constants $A \in \mathbb{Z}, A < 0$ and $B \in \mathbb{Z}, B > 0$ and let us set

$$\tau_{S,[B,\infty)} = \inf\{n \in \mathbb{N} : S_n \geq B\},$$
$$\tau_{S,(A,B)^C} = \inf\{n \in \mathbb{N} : S_n \notin (A, B)\}.$$

You may assume (without having to prove it) that $\tau_{S,[B,\infty)} < \infty$ and $\tau_{S,(A,B)^C} < \infty$ $\mathbb{P}$-a.s..

1) Show that it *doesn't* hold $\mathbb{E}S_0 = \mathbb{E}S_{\tau_{S,[B,\infty)}}$.
2) Recall the statement of optional sampling theorem.
3) Why can't we use optional sampling theorem for point 1.)?
4) Show that $0 = \mathbb{E}S_0 = \mathbb{E}S_{\tau_{S,(A,B)^C}}$.
5) Compute $\mathbb{P}(S_{\tau_{S,(A,B)^C}} = A)$ and $\mathbb{P}(S_{\tau_{S,(A,B)^C}} = B)$.

Exercise 4

Recall martingale convergence theorems and martingale inequalities.